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Properties of the complex bimatrix variate beta distribution

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ABSTRACT

In this article, we derive several properties such as marginal distribution, moments involving zonal polynomials, and asymptotic expansion of the complex bimatrix variate beta type 1 distribution introduced by Díaz-García and Gutiérrez Jáimez [José A. Díaz-García, Ramón Gutiérrez Jáimez, Complex bimatrix variate generalised beta distributions, Linear Algebra Appl. 432 (2010) 571–582]. We also derive distributions of several matrix valued functions of random matrices jointly distributed as complex bimatrix variate beta type 1.

1. Introduction

Let $\boldsymbol{X}$ be an $m \times m$ random Hermitian positive definite matrix such that all its eigenvalues are in the open interval $(0, 1)$. Then, $\boldsymbol{X}$ is said to have a complex matrix variate beta type 1 distribution with parameters (a, b), denoted as $\boldsymbol{X} \sim \mathrm{CB1}(m, a, b)$, if its p.d.f. is given by

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$$\frac{|\boldsymbol{X}|^{a-m}|\boldsymbol{I}_m-\boldsymbol{X}|^{b-m}}{\tilde{B}_m(a,b)}, \tag{1}$$

where $a > m-1$ and $b > m-1$. The complex multivariate beta function, $\tilde{B}_m(a,b)$, used in the above density, is defined as

$$\begin{aligned}\tilde{B}_m(a,b) &= \int_{\boldsymbol{0}<\boldsymbol{Y}=\boldsymbol{Y}^H<\boldsymbol{I}_m} |\boldsymbol{Y}|^{a-m}|\boldsymbol{I}_m-\boldsymbol{Y}|^{b-m}\,\mathrm{d}\boldsymbol{Y}\\ &= \frac{\tilde{\Gamma}_m(a)\tilde{\Gamma}_m(b)}{\tilde{\Gamma}_m(a+b)} = \tilde{B}_m(b,a), \quad \mathrm{Re}(a,b) > m-1, \end{aligned}\tag{2}$$

where the complex multivariate gamma function, $\tilde{\Gamma}_m(c)$, is defined by

$$\begin{aligned}\tilde{\Gamma}_m(c) &= \int_{\boldsymbol{Y}=\boldsymbol{Y}^H>\boldsymbol{0}} |\boldsymbol{Y}|^{c-m}\,\mathrm{etr}(-\boldsymbol{Y})\,\mathrm{d}\boldsymbol{Y}\\ &= \pi^{m(m-1)/2}\prod_{i=1}^{m}\Gamma\left(c-i+1\right), \quad \mathrm{Re}(c) > m-1. \end{aligned}\tag{3}$$

The complex matrix variate beta distribution can be derived by using independent complex gamma matrices. An $m \times m$ random Hermitian positive definite matrix $\boldsymbol{A}$ is said to have a complex gamma distribution with parameters m, ν, and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^H > \boldsymbol{0}$, written as $\boldsymbol{A} \sim \mathrm{CG}(m,\nu,\Sigma)$, if its p.d.f. is given by

$$\frac{|\boldsymbol{A}|^{\nu-m}\,\mathrm{etr}(-\boldsymbol{\Sigma}^{-1}\boldsymbol{A})}{\tilde{\Gamma}_m(\nu)|\boldsymbol{\Sigma}|^{\nu}}, \quad \boldsymbol{A}=\boldsymbol{A}^H>\boldsymbol{0}, \quad \nu \geqslant m. \tag{4}$$

If $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are independent, $\boldsymbol{A}_1 \sim \mathrm{CG}(m,\nu_1,\boldsymbol{I}_m)$ and $\boldsymbol{A}_2 \sim \mathrm{CG}(m,\nu_2,\boldsymbol{I}_m)$, then $\boldsymbol{A}=\boldsymbol{A}_1+\boldsymbol{A}_2$ and $\boldsymbol{U}=\boldsymbol{A}^{-\frac{1}{2}}\boldsymbol{A}_1\boldsymbol{A}^{-\frac{1}{2}}$ are independent, $\boldsymbol{A} \sim \mathrm{CG}(m,\nu_1+\nu_2,\boldsymbol{I}_m)$ and $\boldsymbol{U} \sim \mathrm{CB1}(m,\nu_1,\nu_2)$ (Khatri [16], Gupta [10], and Gupta and Nagar [11–13]).

The complex matrix variate beta distribution arises in various problems in multivariate statistical analysis. Several test statistics in multivariate analysis of variance and covariance are functions of the beta matrix. The complex matrix variate distributions play an important role in various fields of research. Applications of complex random matrices can be found in multiple time series analysis, nuclear physics and radio communications. A number of results on the distribution of the complex random matrices have also been derived. Distributional results on Gaussian, Wishart, Cauchy, beta, and Dirichlet can be found in Tan [24], and Nagar and Arias [19].

Let $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ and $\boldsymbol{A}_3$ be independent complex random matrices, $\boldsymbol{A}_i \sim \mathrm{CG}(m,a_i,I_m)$, $i=1,2,3$, and define

$$\boldsymbol{W}_1 = (\boldsymbol{A}_1+\boldsymbol{A}_3)^{-\frac{1}{2}}\boldsymbol{A}_1(\boldsymbol{A}_1+\boldsymbol{A}_3)^{-\frac{1}{2}} \quad \text{and} \quad \boldsymbol{W}_2 = (\boldsymbol{A}_2+\boldsymbol{A}_3)^{-\frac{1}{2}}\boldsymbol{A}_2(\boldsymbol{A}_2+\boldsymbol{A}_3)^{-\frac{1}{2}}.$$

From the construction, it is easy to see that $\boldsymbol{W}_1 \sim \mathrm{CB1}(m,a_1,a_3)$ and $\boldsymbol{W}_2 \sim \mathrm{CB1}(m,a_2,a_3)$. However, they are correlated so that the joint distribution of complex random matrices $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ is complex bimatrix variate beta type 1. Recently, Díaz-García and Gutiérrez Jáimez [7] have shown that the joint density of $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ is given by

$$\frac{|\boldsymbol{W}_1|^{a_1-m}|\boldsymbol{W}_2|^{a_2-m}|\boldsymbol{I}_m-\boldsymbol{W}_1|^{a_2+a_3-m}|\boldsymbol{I}_m-\boldsymbol{W}_2|^{a_1+a_3-m}}{\tilde{B}_m(a_1,a_2,a_3)|\boldsymbol{I}_m-\boldsymbol{W}_1\boldsymbol{W}_2|^{a_1+a_2+a_3}}, \tag{5}$$

where $\boldsymbol{0} < \boldsymbol{W}_1 = \boldsymbol{W}_1^H < \boldsymbol{I}_m$, $\boldsymbol{0} < \boldsymbol{W}_2 = \boldsymbol{W}_2^H < \boldsymbol{I}_m$ and

$$\tilde{B}_m(a_1,a_2,a_3) = \frac{\tilde{\Gamma}_m(a_1)\tilde{\Gamma}_m(a_2)\tilde{\Gamma}_m(a_3)}{\tilde{\Gamma}_m(a_1+a_2+a+3)}.$$

The complex bimatrix variate beta type 1 distribution defined by the density (5) will be designated as $(\boldsymbol{W}_1,\boldsymbol{W}_2) \sim \mathrm{CB1}(m,a_1,a_2;a_3)$. For $m=1$, the complex matrices $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ are real positive scalars denoted by W_1 and W_2, respectively, and the above density slides to (Libby and Novic [17], Chen and Novic [4], Olkin and Liu [23], and Nagar and Rada-Mora [21]),

$$\frac{w_1^{a_1-1}w_2^{a_2-1}(1-w_1)^{a_2+a_3-1}(1-w_2)^{a_1+a_3-1}}{B(a_1,a_2,a_3)(1-w_1w_2)^{a_1+a_2+a_3}},\quad 0<w_1,w_2<1, \tag{6}$$

where

$$B(a_1,a_2,a_3)=\frac{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}{\Gamma(a_1+a_2+a_3)}.$$

If (W_1,W_2) has p.d.f. given by (6), then we will write $(W_1,W_2)\sim \mathrm{B1}(a_1,a_2;a_3)$.

Díaz-García and Gutiérrez Jáimez [7] have also derived $E[|\boldsymbol{W}_1|^r|\boldsymbol{W}_2|^s]$, the density of the matrix product $\boldsymbol{W}_2^{\frac{1}{2}}\boldsymbol{W}_1\boldsymbol{W}_2^{\frac{1}{2}}$, and the joint distribution of the eigenvalues of $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$. They have also pointed out that the results obtained by them may be considered a generalization of the Jacobi ensemble for the case in which there are two correlated Jacobi ensembles. This distribution, in the real case, has been studied by Gupta and Nagar [14], Díaz-García and Gutiérrez Jáimez [8,9], and Bekker et al. [3].

The aim of this article is to further study properties of the complex bimatrix beta type 1 distribution defined by the density (5).

In Sections 2–4, we study several properties such as transformation, moments involving zonal polynomials, marginal distribution and give asymptotic expansion of the complex bimatrix variate beta type 1 distribution. Finally, in the appendix, we state some well known definitions and results on hypergeometric function of matrix argument, zonal polynomials and integration.

2. Properties

In this section we give several properties of the complex bimatrix variate beta type 1 distribution defined in Section 1 by extending results derived in Gupta and Nagar [14] and Bekker et al. [3] and supplementing the work done by Díaz-García and Gutiérrez Jáimez [7].

Theorem 2.1. *Let $(\boldsymbol{W}_1,\boldsymbol{W}_2)\sim \mathrm{CB1}(m,a_1,a_2;a_3)$ and $\boldsymbol{U}$ be an $m\times m$ unitary matrix, whose elements are either constants or random variables distributed independent of $(\boldsymbol{W}_1,\boldsymbol{W}_2)$. Then, the distribution of $(\boldsymbol{W}_1,\boldsymbol{W}_2)$ is invariant under the transformation $(\boldsymbol{W}_1,\boldsymbol{W}_2)\to(\boldsymbol{U}\boldsymbol{W}_1\boldsymbol{U}^H,\boldsymbol{U}\boldsymbol{W}_2\boldsymbol{U}^H)$, and is independent of $\boldsymbol{U}$ in the latter case.*

Proof. First, let $\boldsymbol{U}$ be a constant unitary matrix. Substituting $\boldsymbol{Z}_1=\boldsymbol{U}\boldsymbol{W}_1\boldsymbol{U}^H$ and $\boldsymbol{Z}_2=\boldsymbol{U}\boldsymbol{W}_2\boldsymbol{U}^H$ with the Jacobian $J(\boldsymbol{W}_1,\boldsymbol{W}_2\to\boldsymbol{Z}_1,\boldsymbol{Z}_2)=1$ in (5) it is easy to see that $(\boldsymbol{U}\boldsymbol{W}_1\boldsymbol{U}^H,\boldsymbol{U}\boldsymbol{W}_2\boldsymbol{U}^H)\sim \mathrm{CB1}(m,a_1,a_2;a_3)$. If, however, $\boldsymbol{U}$ is a random unitary matrix, then $(\boldsymbol{U}\boldsymbol{W}_1\boldsymbol{U}^H,\boldsymbol{U}\boldsymbol{W}_2\boldsymbol{U}^H)|\boldsymbol{U}\sim \mathrm{CB1}(m,a_1,a_2;a_3)$. Since this distribution does not depend on $\boldsymbol{U}$, $(\boldsymbol{U}\boldsymbol{W}_1\boldsymbol{U}^H,\boldsymbol{U}\boldsymbol{W}_2\boldsymbol{U}^H)\sim \mathrm{CB1}(m,a_1,a_2;a_3)$. □

In the next few theorems, we give relationships between complex matrix variate beta type 1, complex matrix variate beta type 2, complex matrix variate Dirichlet type 2 and the complex bimatrix variate beta type 1 distributions. First, we give definitions of complex matrix variate beta type 2 and complex matrix variate Dirichlet type 2 distributions due to Tan [24].

An $m\times m$ random Hermitian positive definite matrix $\boldsymbol{V}$ is said to have a complex matrix variate beta type 2 distribution with parameters $a_1\ (>m-1)$ and $a_2\ (>m-1)$, denoted as $\boldsymbol{V}\sim \mathrm{CB2}(m,a_1,a_2)$, if its p.d.f. is given by

$$\frac{|\boldsymbol{V}|^{a_1-m}|\boldsymbol{I}_m+\boldsymbol{V}|^{-(a_1+a_2)}}{\tilde{B}_m(a_1,a_2)},\quad \boldsymbol{V}=\boldsymbol{V}^H>\boldsymbol{0}. \tag{7}$$

Note that if $\boldsymbol{U}\sim \mathrm{CB1}(m,a_1,a_2)$, then $(\boldsymbol{I}_m-\boldsymbol{U})^{-\frac{1}{2}}\boldsymbol{U}(\boldsymbol{I}_m-\boldsymbol{U})^{-\frac{1}{2}}\sim \mathrm{CB2}(m,a_1,a_2)$.

The $m \times m$ Hermitian positive definite random matrices $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are said to have a complex Dirichlet type 2 distribution of order three, denoted as $(\boldsymbol{V}_1, \boldsymbol{V}_2) \sim \text{CD2}(m, a_1, a_2; a_3)$, if their joint p.d.f. is given by

$$\frac{|\boldsymbol{V}_1|^{a_1-m}|\boldsymbol{V}_2|^{a_2-m}|\boldsymbol{I}_m + \boldsymbol{V}_1 + \boldsymbol{V}_2|^{-(a_1+a_2+a_3)}}{\tilde{B}_m(a_1, a_2, a_3)}, \quad \boldsymbol{V}_1 = \boldsymbol{V}_1^H > \boldsymbol{0}, \quad \boldsymbol{V}_2 = \boldsymbol{V}_2^H > \boldsymbol{0}, \tag{8}$$

where $a_i > m - 1, i = 1, 2, 3$. Note that, this distribution is a special case of the complex matrix variate Dirichlet type 2 distribution.

Theorem 2.2. *Let* $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim \text{CB1}(m, a_1, a_2; a_3)$. *Then,* $\boldsymbol{W}_1 \sim \text{CB1}(m, a_1, a_3)$, $\boldsymbol{W}_2 \sim \text{CB1}(m, a_2, a_3)$, $(\boldsymbol{I}_m - \boldsymbol{W}_1)^{-\frac{1}{2}}\boldsymbol{W}_1(\boldsymbol{I}_m - \boldsymbol{W}_1)^{-\frac{1}{2}} \sim \text{CB2}(m, a_1, a_3)$ *and* $(\boldsymbol{I}_m - \boldsymbol{W}_2)^{-\frac{1}{2}}\boldsymbol{W}_2(\boldsymbol{I}_m - \boldsymbol{W}_2)^{-\frac{1}{2}} \sim \text{CB2}(m, a_2, a_3)$.

Proof. Integrating $\boldsymbol{W}_2$ in (5) using (A.5), and simplifying the resulting expression using (A.3), the marginal density of $\boldsymbol{W}_1$ is obtained. $\square$

Theorem 2.3. *Let* $(\boldsymbol{V}_1, \boldsymbol{V}_2) \sim \text{CD2}(m, a_1, a_2; a_3)$ *and define* $\boldsymbol{W}_1 = (\boldsymbol{I}_m + \boldsymbol{V}_1)^{-\frac{1}{2}}\boldsymbol{V}_1(\boldsymbol{I}_m + \boldsymbol{V}_1)^{-\frac{1}{2}} = \boldsymbol{I}_m - (\boldsymbol{I}_m + \boldsymbol{V}_1)^{-1}$ *and* $\boldsymbol{W}_2 = (\boldsymbol{I}_m + \boldsymbol{V}_2)^{-\frac{1}{2}}\boldsymbol{V}_2(\boldsymbol{I}_m + \boldsymbol{V}_2)^{-\frac{1}{2}} = \boldsymbol{I}_m - (\boldsymbol{I}_m + \boldsymbol{V}_2)^{-1}$. *Then,* $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim \text{CB1}(m, a_1, a_2; a_3)$.

Proof. Making the transformation $\boldsymbol{W}_i = \boldsymbol{I}_m - (\boldsymbol{I}_m + \boldsymbol{V}_i)^{-1}, i = 1, 2$ with the Jacobian $J(\boldsymbol{V}_1, \boldsymbol{V}_2 \to \boldsymbol{W}_1, \boldsymbol{W}_2) = |\boldsymbol{I}_m - \boldsymbol{W}_1|^{-2m}|\boldsymbol{I}_m - \boldsymbol{W}_2|^{-2m}$ in (8) and simplifying we get the desired result. $\square$

Theorem 2.4. *Let* $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim \text{CB1}(m, a_1, a_2; a_3)$. *Define* $\boldsymbol{V}_1 = (\boldsymbol{I}_m - \boldsymbol{W}_1)^{-\frac{1}{2}}\boldsymbol{W}_1(\boldsymbol{I}_m - \boldsymbol{W}_1)^{-\frac{1}{2}} = (\boldsymbol{I}_m - \boldsymbol{W}_1)^{-1} - \boldsymbol{I}_m$ *and* $\boldsymbol{V}_2 = (\boldsymbol{I}_m - \boldsymbol{W}_2)^{-\frac{1}{2}}\boldsymbol{W}_2(\boldsymbol{I}_m - \boldsymbol{W}_2)^{-\frac{1}{2}} = (\boldsymbol{I}_m - \boldsymbol{W}_2)^{-1} - \boldsymbol{I}_m$. *Then,* $(\boldsymbol{V}_1, \boldsymbol{V}_2) \sim \text{CD2}(m, a_1, a_2; a_3)$.

Proof. Transforming $\boldsymbol{V}_i = (\boldsymbol{I}_m - \boldsymbol{W}_i)^{-1} - \boldsymbol{I}_m, i = 1, 2$ with the Jacobian $J(\boldsymbol{W}_1, \boldsymbol{W}_2 \to \boldsymbol{V}_1, \boldsymbol{V}_2) = |\boldsymbol{I}_m + \boldsymbol{V}_1|^{-2m}|\boldsymbol{I}_m + \boldsymbol{V}_2|^{-2m}$ in (5) and simplifying we get the desired result. $\square$

Theorem 2.5. *Let* $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim \text{CB1}(m, a_1, a_2; a_3)$. *Define* $\boldsymbol{X}_1 = \boldsymbol{W}_1^{\frac{1}{2}}\boldsymbol{W}_2\boldsymbol{W}_1^{\frac{1}{2}}$ *and* $\boldsymbol{Y}_1 = (\boldsymbol{I}_m - \boldsymbol{W}_1^{\frac{1}{2}}\boldsymbol{W}_2\boldsymbol{W}_1^{\frac{1}{2}})^{-\frac{1}{2}}(\boldsymbol{I}_m - \boldsymbol{W}_1)(\boldsymbol{I}_m - \boldsymbol{W}_1^{\frac{1}{2}}\boldsymbol{W}_2\boldsymbol{W}_1^{\frac{1}{2}})^{-\frac{1}{2}}$. *Then,* $\boldsymbol{Y}_1 \sim \text{CB1}(m, a_2 + a_3, a_1)$ *and the marginal density of* $\boldsymbol{X}_1$ *is given by*

$$\begin{aligned}&\frac{\tilde{B}_m(a_2 + a_3, a_1 + a_3)}{\tilde{B}_m(a_1, a_2, a_3)}|\boldsymbol{X}_1|^{a_2-m}|\boldsymbol{I}_m - \boldsymbol{X}_1|^{a_3-m} \\ &\quad \times {}_2\tilde{F}_1(a_2 + a_3, a_2 + a_3; a_1 + a_2 + 2a_3; \boldsymbol{I}_m - \boldsymbol{X}_1), \quad \boldsymbol{0} < \boldsymbol{X}_1 = \boldsymbol{X}_1^H < \boldsymbol{I}_m.\end{aligned}$$

Proof. Transforming $\boldsymbol{X}_1 = \boldsymbol{W}_1^{\frac{1}{2}}\boldsymbol{W}_2\boldsymbol{W}_1^{\frac{1}{2}}$ and $\boldsymbol{Y}_1 = (\boldsymbol{I}_m - \boldsymbol{X}_1)^{-\frac{1}{2}}(\boldsymbol{I}_m - \boldsymbol{W}_1)\ (\boldsymbol{I}_m - \boldsymbol{X}_1)^{-\frac{1}{2}}$ with the Jacobian $J(\boldsymbol{W}_1, \boldsymbol{W}_2 \to \boldsymbol{X}_1, \boldsymbol{Y}_1) = |\boldsymbol{I}_m - \boldsymbol{X}_1|^m\ |\boldsymbol{I}_m - (\boldsymbol{I}_m - \boldsymbol{X}_1)\boldsymbol{Y}_1|^{-m}$ in (5), the joint density of $\boldsymbol{X}_1$ and $\boldsymbol{Y}_1$ is derived as

$$\frac{|\boldsymbol{X}_1|^{a_2-m}|\boldsymbol{Y}_1|^{a_2+a_3-m}|\boldsymbol{I}_m - \boldsymbol{X}_1|^{a_3-m}|\boldsymbol{I}_m - \boldsymbol{Y}_1|^{a_1+a_3-m}}{\tilde{B}_m(a_1, a_2, a_3)|\boldsymbol{I}_m - (\boldsymbol{I}_m - \boldsymbol{X}_1)\boldsymbol{Y}_1|^{a_2+a_3}}, \tag{9}$$

where $\boldsymbol{0} < \boldsymbol{X}_1 = \boldsymbol{X}_1^H < \boldsymbol{I}_m$ and $\boldsymbol{0} < \boldsymbol{Y}_1 = \boldsymbol{Y}_1^H < \boldsymbol{I}_m$. Now, integrating the above density appropriately, using

$$\begin{aligned}\int_{\boldsymbol{0}<\boldsymbol{X}_1=\boldsymbol{X}_1^H<\boldsymbol{I}_m} \frac{|\boldsymbol{X}_1|^{a_2-m}|\boldsymbol{I}_m - \boldsymbol{X}_1|^{a_3-m}\,\mathrm{d}\boldsymbol{X}_1}{|\boldsymbol{I}_m - (\boldsymbol{I}_m - \boldsymbol{X}_1)\boldsymbol{Y}_1|^{a_2+a_3}} &= \tilde{B}_m(a_2, a_3)\,{}_2\tilde{F}_1(a_3, a_2 + a_3; a_2 + a_3; \boldsymbol{Y}_1) \\ &= \tilde{B}_m(a_2, a_3)\,{}_1\tilde{F}_0(a_3; \boldsymbol{Y}_1) = \tilde{B}_m(a_2, a_3)|\boldsymbol{I}_m - \boldsymbol{Y}_1|^{-a_3}\end{aligned}$$

and

$$\int_{\mathbf{0}<\boldsymbol{Y}_1=\boldsymbol{Y}_1^H<\boldsymbol{I}_m} \frac{|\boldsymbol{Y}_1|^{a_2+a_3-m}|\boldsymbol{I}_m-\boldsymbol{Y}_1|^{a_1+a_3-m}\,\mathrm{d}\boldsymbol{Y}_1}{|\boldsymbol{I}_m-(\boldsymbol{I}_m-\boldsymbol{X}_1)\boldsymbol{Y}_1|^{a_2+a_3}}$$
$$= \tilde{B}_m(a_2+a_3, a_1+a_3)\,{}_2\tilde{F}_1(a_2+a_3, a_2+a_3; a_1+a_2+2a_3; \boldsymbol{I}_m-\boldsymbol{X}_1),$$

we get the desired result. $\square$

Let $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim \mathrm{CB1}(m, a_1, a_2; a_3)$. Define $\boldsymbol{X}_2 = \boldsymbol{W}_2^{\frac{1}{2}}\boldsymbol{W}_1\boldsymbol{W}_2^{\frac{1}{2}}$,

$$\boldsymbol{Y}_2 = \left(\boldsymbol{I}_m - \boldsymbol{W}_2^{\frac{1}{2}}\boldsymbol{W}_1\boldsymbol{W}_2^{\frac{1}{2}}\right)^{-\frac{1}{2}}(\boldsymbol{I}_m-\boldsymbol{W}_2)\left(\boldsymbol{I}_m - \boldsymbol{W}_2^{\frac{1}{2}}\boldsymbol{W}_1\boldsymbol{W}_2^{\frac{1}{2}}\right)^{-\frac{1}{2}},$$
$$\boldsymbol{Z}_1 = \left(\boldsymbol{I}_m - \boldsymbol{W}_1^{\frac{1}{2}}\boldsymbol{W}_2\boldsymbol{W}_1^{\frac{1}{2}}\right)^{-\frac{1}{2}}\boldsymbol{W}_1^{\frac{1}{2}}(\boldsymbol{I}_m-\boldsymbol{W}_2)\boldsymbol{W}_1^{\frac{1}{2}}\left(\boldsymbol{I}_m - \boldsymbol{W}_1^{\frac{1}{2}}\boldsymbol{W}_2\boldsymbol{W}_1^{\frac{1}{2}}\right)^{-\frac{1}{2}},$$

and

$$\boldsymbol{Z}_2 = \left(\boldsymbol{I}_m - \boldsymbol{W}_2^{\frac{1}{2}}\boldsymbol{W}_1\boldsymbol{W}_2^{\frac{1}{2}}\right)^{-\frac{1}{2}}\boldsymbol{W}_2^{\frac{1}{2}}(\boldsymbol{I}_m-\boldsymbol{W}_1)\boldsymbol{W}_2^{\frac{1}{2}}\left(\boldsymbol{I}_m - \boldsymbol{W}_2^{\frac{1}{2}}\boldsymbol{W}_1\boldsymbol{W}_2^{\frac{1}{2}}\right)^{-\frac{1}{2}}.$$

Then, from the above theorem, it can easily be deduced that $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ have identical distributions, $\boldsymbol{Y}_2 \sim \mathrm{CB1}(m, a_1+a_3, a_2)$, $\boldsymbol{Z}_1 = \boldsymbol{I}_m - \boldsymbol{Y}_1 \sim \mathrm{CB1}(m, a_1, a_2+a_3)$ and $\boldsymbol{Z}_2 = \boldsymbol{I}_m - \boldsymbol{Y}_2 \sim \mathrm{CB1}(m, a_2, a_1+a_3)$.

Using the definition and effecting the integration using (A.5) and (A.7), the joint (r, s)th moment of $|\boldsymbol{W}_1|$ and $|\boldsymbol{W}_2|$ is obtained as

$$E[|\boldsymbol{W}_1|^r|\boldsymbol{W}_2|^s]$$
$$= \frac{\tilde{B}_m(a_2+s, a_1+a_3)\tilde{B}_m(a_1+r, a_2+a_3)}{\tilde{B}_m(a_1, a_2, a_3)}$$
$$\times\, {}_3\tilde{F}_2(a_1+r, a_2+s, a_1+a_2+a_3; a_1+a_2+a_3+r, a_1+a_2+a_3+s; \boldsymbol{I}_m).$$

Substituting $r = s = h$ above, expanding ${}_3\tilde{F}_2$ by applying (A.1) and simplifying the resulting expression using (3) and (A.2), the hth moment of $P = |\boldsymbol{W}_1\boldsymbol{W}_2|$ is obtained as

$$E(P^h) = \frac{\tilde{\Gamma}_m(a_1+a_3)\tilde{\Gamma}_m(a_2+a_3)}{\tilde{B}_m(a_1, a_2, a_3)}\sum_{k=0}^{\infty}\sum_{\kappa\vdash k}\frac{[a_1+a_2+a_3]_\kappa}{k!}$$
$$\times\left\{\prod_{i=1}^{m}\frac{\Gamma(a_1+h-i+1+k_i)\Gamma(a_2+h-i+1+k_i)}{[\Gamma(a_1+a_2+a_3+h-i+1+k_i)]^2}\right\}\tilde{C}_\kappa(\boldsymbol{I}_m).$$

Now, using the inverse Mellin transform and $E(P^h)$, the density function of P is derived as

$$\frac{\tilde{\Gamma}_m(a_1+a_3)\tilde{\Gamma}_m(a_2+a_3)}{\tilde{B}_m(a_1, a_2, a_3)}p^{-1}\sum_{k=0}^{\infty}\sum_{\kappa\vdash k}\frac{[a_1+a_2+a_3]_\kappa}{k!}\tilde{C}_\kappa(\boldsymbol{I}_m)$$
$$\times\, G^{2m,0}_{2m,2m}\left[p\,\middle|\,\begin{matrix}(a_1+a_2+a_3-i+1+k_i, a_1+a_2+a_3-i+1+k_i), i=1,\ldots,m\\ (a_1-i+1+k_i, a_2-i+1+k_i), i=1,\ldots,m\end{matrix}\right],$$

where $0 < p < 1$ and $G^{m,n}_{p,q}$ is the Meijer's G-function. By substituting $m = 1$ and using the results on G-function (Luke [18] and Bagai [2]),

$$G^{2,0}_{2,2}\left[z \middle| \begin{matrix} a_1 + b_1 - 1, a_2 + b_2 - 1 \\ a_1 - 1, a_2 - 1 \end{matrix}\right]$$
$$= \frac{z^{a_2-1}(1-z)^{b_1+b_2-1}}{\Gamma(b_1+b_2)} {}_2F_1(a_2 + b_2 - a_1, b_1; b_1 + b_2; 1 - z), \ |z| < 1,$$

the density of $P = W_1 W_2$, where $(W_1, W_2) \sim \mathrm{CB1}(a_1, a_2; a_3)$, is derived as

$$\frac{B(a_2 + a_3, a_1 + a_3)}{B(a_1, a_2, a_3)} p^{a_2-1}(1-p)^{a_3-1}$$
$$\times {}_2\tilde{F}_1(a_2 + a_3, a_2 + a_3; a_1 + a_2 + 2a_3; 1 - p), \quad 0 < p < 1.$$

The above density is also derived in Nagar et al. [22].

Further, $E[\tilde{C}_\kappa(\boldsymbol{W}_1\boldsymbol{W}_2)]$ is derived as

$$E[\tilde{C}_\kappa(\boldsymbol{W}_1\boldsymbol{W}_2)] = \int_{\boldsymbol{0}<\boldsymbol{W}_2=\boldsymbol{W}_2^H<\boldsymbol{I}_m} \int_{\boldsymbol{0}<\boldsymbol{W}_2=\boldsymbol{W}_2^H<\boldsymbol{I}_m} \frac{\tilde{C}_\kappa(\boldsymbol{W}_1\boldsymbol{W}_2)|\boldsymbol{W}_1|^{a_1-m}|\boldsymbol{W}_2|^{a_2-m}}{\tilde{B}_m(a_1, a_2, a_3)}$$
$$\times \frac{|\boldsymbol{I}_m - \boldsymbol{W}_1|^{a_2+a_3-m}|\boldsymbol{I}_m - \boldsymbol{W}_2|^{a_1+a_3-m} \, d\boldsymbol{W}_1 \, d\boldsymbol{W}_2}{|\boldsymbol{I}_m - \boldsymbol{W}_1\boldsymbol{W}_2|^{a_1+a_2+a_3}}.$$

For evaluating the above integral, we use (A.3), (A.6) and (A.8) obtaining

$$E[\tilde{C}_\kappa(\boldsymbol{W}_1\boldsymbol{W}_2)] = \frac{\tilde{\Gamma}_m(a_2+a_3)\tilde{\Gamma}_m(a_1+a_3)}{\tilde{\Gamma}_m(a_1+a_2+a_3)\tilde{\Gamma}_m(a_3)} \sum_{t=0}^{\infty}\sum_{\tau\vdash t} \frac{(a_1+a_2+a_3)_\tau}{t!}$$
$$\times \sum_{\delta\vdash t+k} \frac{(a_1)_\delta (a_2)_\delta}{[(a_1+a_2+a_3)_\delta]^2} \tilde{C}_\delta(\boldsymbol{I}_m).$$

An alternative expression in terms of invariant polynomials can also be derived using the density of $\boldsymbol{W}_1^{\frac{1}{2}}\boldsymbol{W}_2\boldsymbol{W}_1^{\frac{1}{2}}$ and results on invariant polynomials.

3. Marginal distribution

Let $(\boldsymbol{V}_1, \boldsymbol{V}_2) \sim \mathrm{CD2}(m, a_1, a_2; a_3)$ and

$$\boldsymbol{V}_i = \begin{pmatrix} \boldsymbol{V}_{11(i)} & \boldsymbol{V}_{12(i)} \\ \boldsymbol{V}_{21(i)} & \boldsymbol{V}_{22(i)} \end{pmatrix}, \quad \boldsymbol{V}_{11(i)} \ (q \times q)$$

and $\boldsymbol{V}_{11\cdot 2(i)} = \boldsymbol{V}_{11(i)} - \boldsymbol{V}_{12(i)}\boldsymbol{V}_{22(i)}^{-1}\boldsymbol{V}_{21(i)}$, $i = 1, 2$. It is well known that $(\boldsymbol{V}_{11\cdot 2(1)}, \boldsymbol{V}_{11\cdot 2(2)})$ and $(\boldsymbol{V}_{22(1)}, \boldsymbol{V}_{22(2)})$ are distributed independently (Gupta and Nagar [13], Nagar and Bedoya [20], and Tan [24]), $(\boldsymbol{V}_{11\cdot 2(1)}, \boldsymbol{V}_{11\cdot 2(2)}) \sim \mathrm{CD2}(q, a_1 - m + q, a_2 - m + q; a_3)$ and $(\boldsymbol{V}_{22(1)}, \boldsymbol{V}_{22(2)}) \sim \mathrm{CD2}(m - q, a_1, a_2; a_3 - q)$.

In this section we derive similar result for the complex bimatrix variate beta type 1 distribution defined in Section 1.

Theorem 3.1. *Let* $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim \mathrm{CB1}(m, a_1, a_2; a_3)$ *and partition* $\boldsymbol{W}_i$ *as*

$$\boldsymbol{W}_i = \begin{pmatrix} \boldsymbol{W}_{11(i)} & \boldsymbol{W}_{12(i)} \\ \boldsymbol{W}_{21(i)} & \boldsymbol{W}_{22(i)} \end{pmatrix}, \quad \boldsymbol{W}_{11(i)} \ (q \times q).$$

Then, $(\boldsymbol{W}_{11\cdot 2(1)}, \boldsymbol{W}_{11\cdot 2(2)}) \sim \text{CB1}(q, a_1 - m + q, a_2 - m + q; a_3)$, *where* $\boldsymbol{W}_{11\cdot 2(i)} = \boldsymbol{W}_{11(i)} - \boldsymbol{W}_{12(i)}\boldsymbol{W}_{22(i)}^{-1}\boldsymbol{W}_{21(i)}$, $i = 1, 2$.

Proof. According to the Theorem 2.3, if $(\boldsymbol{V}_1, \boldsymbol{V}_2) \sim \text{CD2}(m, a_1, a_2; a_3)$, $\boldsymbol{W}_1 = (\boldsymbol{I}_m + \boldsymbol{V}_1)^{-\frac{1}{2}}\boldsymbol{V}_1(\boldsymbol{I}_m + \boldsymbol{V}_1)^{-\frac{1}{2}}$ and $\boldsymbol{W}_2 = (\boldsymbol{I}_m + \boldsymbol{V}_2)^{-\frac{1}{2}}\boldsymbol{V}_2(\boldsymbol{I}_m + \boldsymbol{V}_2)^{-\frac{1}{2}}$, then, $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim \text{CB1}(m, a_1, a_2; a_3)$. Further, if

$$\boldsymbol{W}^{-1} = \begin{pmatrix} \boldsymbol{W}^{11} & \boldsymbol{W}^{12} \\ \boldsymbol{W}^{21} & \boldsymbol{W}^{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_q + \boldsymbol{V}^{11} & \boldsymbol{V}^{12} \\ \boldsymbol{V}^{21} & \boldsymbol{I}_{m-q} + \boldsymbol{V}^{22} \end{pmatrix}$$

then $\boldsymbol{W}^{11}\ (q \times q) = \boldsymbol{W}_{11\cdot 2}^{-1} = \boldsymbol{I}_q + \boldsymbol{V}^{11} = \boldsymbol{I}_q + \boldsymbol{V}_{11\cdot 2}^{-1}$ and $\boldsymbol{W}_{11\cdot 2} = (\boldsymbol{I}_q + \boldsymbol{V}_{11\cdot 2})^{-1}\boldsymbol{V}_{11\cdot 2} = \boldsymbol{I}_m - (\boldsymbol{I}_m + \boldsymbol{V}_{11\cdot 2})^{-1}$. Now, since $(\boldsymbol{V}_{11\cdot 2(1)}, \boldsymbol{V}_{11\cdot 2(2)}) \sim \text{CD2}(q, a_1 - m + q, a_2 - m + q; a_3)$, we have from Theorem 2.3, $(\boldsymbol{W}_{11\cdot 2(1)}, \boldsymbol{W}_{11\cdot 2(2)}) \sim \text{CB1}(q, a_1 - m + q, a_2 - m + q; a_3)$. □

The distribution of $((\boldsymbol{C}\boldsymbol{W}_1^{-1}\boldsymbol{C}^H)^{-1}, (\boldsymbol{C}\boldsymbol{W}_2^{-1}\boldsymbol{C}^H)^{-1})$ where $\boldsymbol{C}\ (q \times m)$ is a constant matrix of rank $q\ (\leqslant m)$, is now derived.

Theorem 3.2. *Let* $\boldsymbol{C}$ *be a* $q \times m$ *complex non-random matrix of rank* $q\ (\leqslant m)$. *If* $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim \text{CB1}(m, a_1, a_2; a_3)$, *then*

$$\begin{aligned} &\left((\boldsymbol{C}\boldsymbol{C}^H)^{\frac{1}{2}}(\boldsymbol{C}\boldsymbol{W}_1^{-1}\boldsymbol{C}^H)^{-1}(\boldsymbol{C}\boldsymbol{C}^H)^{\frac{1}{2}}, (\boldsymbol{C}\boldsymbol{C}^H)^{\frac{1}{2}}(\boldsymbol{C}\boldsymbol{W}_1^{-1}\boldsymbol{C}^H)^{-1}(\boldsymbol{C}\boldsymbol{C}^H)^{\frac{1}{2}}\right) \\ &\sim \text{CB1}(q, a_1 - m + q, a_2 - m + q; a_3), \end{aligned}$$

Proof. Write $\boldsymbol{C} = \boldsymbol{M}\begin{pmatrix}\boldsymbol{I}_q & 0\end{pmatrix}\boldsymbol{G}$, where $\boldsymbol{M}\ (q \times q)$ is Hermitian positive definite and $\boldsymbol{G}\ (m \times m)$ is unitary. Now, for $i = 1, 2$,

$$\begin{aligned} (\boldsymbol{C}\boldsymbol{W}_i^{-1}\boldsymbol{C}^H)^{-1} &= \left[\boldsymbol{M}\begin{pmatrix}\boldsymbol{I}_q & \boldsymbol{0}\end{pmatrix}\boldsymbol{G}\boldsymbol{W}_i^{-1}\boldsymbol{G}^H\begin{pmatrix}\boldsymbol{I}_q & \boldsymbol{0}\end{pmatrix}^H\boldsymbol{M}^H\right]^{-1} \\ &= (\boldsymbol{M}^H)^{-1}\left[\begin{pmatrix}\boldsymbol{I}_q & \boldsymbol{0}\end{pmatrix}\boldsymbol{U}_i^{-1}\begin{pmatrix}\boldsymbol{I}_q \\ \boldsymbol{0}\end{pmatrix}\right]^{-1}\boldsymbol{M}^{-1} \\ &= (\boldsymbol{M}^H)^{-1}(\boldsymbol{U}_i^{11})^{-1}\boldsymbol{M}^{-1}, \end{aligned}$$

where

$$\boldsymbol{U}_i = \boldsymbol{G}\boldsymbol{W}_i\boldsymbol{G}^H = \begin{pmatrix} \boldsymbol{U}_{11(i)} & \boldsymbol{U}_{12(i)} \\ \boldsymbol{U}_{21(i)} & \boldsymbol{U}_{22(i)} \end{pmatrix}, \quad \boldsymbol{U}_{11(i)}\ (q \times q),$$

and $\boldsymbol{U}_i^{11} = \boldsymbol{U}_{11\cdot 2(i)}^{-1}$. Note that $(\boldsymbol{U}_1, \boldsymbol{U}_2) \sim \text{CB1}(m, a_1, a_2; a_3)$. Hence, from Theorem 3.1, $(\boldsymbol{U}_{11\cdot 2(1)}, \boldsymbol{U}_{11\cdot 2(2)}) \sim \text{CB1}(q, a_1 - m + q, a_2 - m + q; a_3)$. Now, noting that $\boldsymbol{M}\boldsymbol{M}^H = \boldsymbol{C}\boldsymbol{C}^H$ and $(\boldsymbol{M}\boldsymbol{M}^H)^{\frac{1}{2}}(\boldsymbol{M}^H)^{-1}$ is a unitary matrix of order q, we have

$$\begin{aligned} &(\boldsymbol{C}\boldsymbol{C}^H)^{\frac{1}{2}}\left((\boldsymbol{C}\boldsymbol{W}_1^{-1}\boldsymbol{C}^H)^{-1}, (\boldsymbol{C}\boldsymbol{W}_1^{-1}\boldsymbol{C}^H)^{-1}\right)(\boldsymbol{C}\boldsymbol{C}^H)^{\frac{1}{2}} \\ &\quad = (\boldsymbol{M}\boldsymbol{M}^H)^{\frac{1}{2}}(\boldsymbol{M}^H)^{-1}(\boldsymbol{U}_{11\cdot 2(1)}, \boldsymbol{U}_{11\cdot 2(2)})\boldsymbol{M}^{-1}(\boldsymbol{M}\boldsymbol{M}^H)^{\frac{1}{2}} \\ &\quad \sim \text{CB1}(q, a_1 - m + q, a_2 - m + q; a_3), \end{aligned}$$

which is the desired result. □

From the above theorem, when $\boldsymbol{C} \equiv \mathbf{c} \in \mathbb{C}^m$, $\mathbf{c} \neq \mathbf{0}$, it follows that

$$\left(\frac{\mathbf{c}^H\mathbf{c}}{\mathbf{c}^H\boldsymbol{W}_1^{-1}\mathbf{c}}, \frac{\mathbf{c}^H\mathbf{c}}{\mathbf{c}^H\boldsymbol{W}_2^{-1}\mathbf{c}}\right) \sim \mathrm{B1}(a_1 - m + 1, a_2 - m + 1; a_3).$$

Further, if $\mathbf{z}$ $(m \times 1)$ is a complex random vector, and $P(\mathbf{z} \neq \mathbf{0}) = 1$, then

$$\left(\frac{\mathbf{z}^H\mathbf{z}}{\mathbf{z}^H\boldsymbol{W}_1^{-1}\mathbf{z}}, \frac{\mathbf{z}^H\mathbf{z}}{\mathbf{z}^H\boldsymbol{W}_2^{-1}\mathbf{z}}\right) \sim \mathrm{B1}(a_1 - m + 1, a_2 - m + 1; a_3)$$

if $\mathbf{z}$ is independent of $(\boldsymbol{W}_1, \boldsymbol{W}_2)$.

4. Asymptotic expansion

In this section, we derive the asymptotic expansion of the complex bimatrix variate beta type 1 density. We first give two lemmas which are needed to derive the final result.

Lemma 4.1. *For α_1, α_2 scalars, we have*

$$\begin{aligned}\ln\left[\frac{\tilde{\Gamma}_m(z+\alpha_1)}{\tilde{\Gamma}_m(z+\alpha_2)}\right] &= (\alpha_1 - \alpha_2)m\ln z \\ &+ \sum_{i=1}^{m}\sum_{s=1}^{r}\frac{(-1)^{s+1}}{s(s+1)}\left[B_{s+1}\left(\alpha_1 - i + 1\right) - B_{s+1}\left(\alpha_2 - i + 1\right)\right]z^{-s} \\ &+ O(z^{-r-1}), |\arg(z)| \leqslant \pi - \epsilon, \epsilon > 0,\end{aligned}$$

where $B_k(x)$ is the Bernoulli polynomial of degree k and order unity.

Proof. See Gupta et al. [15]. □

Lemma 4.2. *For $||\boldsymbol{Z}/n|| < 1$,*

$$-\ln\det\left(I_m - \frac{\boldsymbol{Z}}{n}\right) = \sum_{s=1}^{r}\frac{n^{-s}\,\mathrm{tr}(\boldsymbol{Z}^s)}{s} + O(n^{-r-1}).$$

Theorem 4.1. *Let $(\boldsymbol{W}_1, \boldsymbol{W}_2) \sim \mathrm{CB1}(m, a_1, a_2; a_3)$. Define $\boldsymbol{Y}_i = a_3\boldsymbol{W}_i$, $i = 1, 2$. Then, the p.d.f. of $(\boldsymbol{Y}_1, \boldsymbol{Y}_2)$ can be expanded as*

$$\begin{aligned}&\left[\frac{\mathrm{etr}\left[-(\boldsymbol{Y}_1 + \boldsymbol{Y}_2)\right]|\boldsymbol{Y}_1|^{a_1-m}|\boldsymbol{Y}_2|^{a_2-m}}{\tilde{\Gamma}_m(a_1)\tilde{\Gamma}_m(a_2)}\right] \\ &\times\left[1 + \frac{\tilde{c}_1}{2a_3} + \frac{3\tilde{c}_1^2 + 4\tilde{c}_2}{24a_3^2} + O(a_3^{-3})\right], \quad \boldsymbol{Y}_1 = \boldsymbol{Y}_1^H > \mathbf{0}, \quad \boldsymbol{Y}_2 = \boldsymbol{Y}_2^H > \mathbf{0}, \end{aligned} \tag{10}$$

where

$$\begin{aligned}\tilde{c}_1 &= -2\,\mathrm{tr}(a_2\boldsymbol{Y}_1 + a_1\boldsymbol{Y}_2) + 2m\,\mathrm{tr}(\boldsymbol{Y}_1 + \boldsymbol{Y}_2) - \mathrm{tr}(\boldsymbol{Y}_1 - \boldsymbol{Y}_2)^2 + am(a - m), \\ \tilde{c}_2 &= -3\,\mathrm{tr}(a_2\boldsymbol{Y}_1^2 + a_1\boldsymbol{Y}_2^2) + 3m\,\mathrm{tr}(\boldsymbol{Y}_1^2 + \boldsymbol{Y}_2^2) - 2\,\mathrm{tr}(\boldsymbol{Y}_1^3 + \boldsymbol{Y}_2^3) \\ &\quad + 6a\,\mathrm{tr}(\boldsymbol{Y}_1\boldsymbol{Y}_2) - \frac{1}{2}am[2a^2 - 3am + m^2 - 1],\end{aligned}$$

and $a = a_1 + a_2$.

Proof. Substituting $\boldsymbol{Y}_i = a_3\boldsymbol{W}_i$, $i = 1, 2$, with $J(\boldsymbol{W}_1, \boldsymbol{W}_2, \to \boldsymbol{Y}_1, \boldsymbol{Y}_2) = a_3^{-2m^2}$ in (5), we obtain the p.d.f. of $(\boldsymbol{Y}_1, \boldsymbol{Y}_2)$ as

$$\frac{|\boldsymbol{Y}_1|^{a_1-m}|\boldsymbol{Y}_2|^{a_2-m}}{\tilde{\Gamma}_m(a_1)\tilde{\Gamma}_m(a_2)}\tilde{I}_1\tilde{I}_2, \quad \boldsymbol{Y}_1 = \boldsymbol{Y}_1^H > \boldsymbol{0}, \quad \boldsymbol{Y}_2 = \boldsymbol{Y}_2^H > \boldsymbol{0}, \tag{11}$$

where

$$\tilde{I}_1 = \frac{\tilde{\Gamma}_m(a_1+a_2+a_3)}{\tilde{\Gamma}_m(a_3)}a_3^{-m(a_1+a_2)},$$

and

$$\tilde{I}_2 = \frac{|\boldsymbol{I}_m - \boldsymbol{Y}_1/a_3|^{a_2+a_3-m}|\boldsymbol{I}_m - \boldsymbol{Y}_2/a_3|^{a_1+a_3-m}}{|\boldsymbol{I}_m - \boldsymbol{Y}_1\boldsymbol{Y}_2/a_3^2|^{a_1+a_2+a_3}}.$$

Using Lemma 4.1 with $r = 2$, $z = a_3$, $\alpha_1 = a_1 + a_2 = a$ and $\alpha_2 = 0$, we obtain

$$\begin{aligned}\ln \tilde{I}_1 = {}& \frac{1}{2a_3}\sum_{i=1}^{m}[B_2\,(a-i+1) - B_2\,(1-i)] \\ & -\frac{1}{6a_3^2}\sum_{i=1}^{m}[B_3\,(a-i+1) - B_3\,(1-i)] + O(a_3^{-3}),\end{aligned}$$

where $B_2(x) = x^2 - x + 1/6$ and $B_3(x) = x^3 - 3x^2/2 + x/2$. Now, substituting for $B_2(\cdot)$ and $B_3(\cdot)$ in the above expression and simplifying, we obtain

$$\ln \tilde{I}_1 = \frac{1}{2a_3}[a^2m - am^2] - \frac{1}{12a_3^2}am[2a^2 - 3am + m^2 - 1] + O\left(a_3^{-3}\right). \tag{12}$$

Further, the application of Lemma 4.2 yields

$$\begin{aligned}\ln \tilde{I}_2 = {}& -\operatorname{tr}\,(\boldsymbol{Y}_1 + \boldsymbol{Y}_2) - \frac{1}{2a_3}[2\operatorname{tr}(a_2\boldsymbol{Y}_1 + a_1\boldsymbol{Y}_2) - 2m\operatorname{tr}(\boldsymbol{Y}_1 + \boldsymbol{Y}_2) \\ & + \operatorname{tr}(\boldsymbol{Y}_1 - \boldsymbol{Y}_2)^2] - \frac{1}{6a_3^2}[3\operatorname{tr}(a_2\boldsymbol{Y}_1^2 + a_1\boldsymbol{Y}_2^2) - 3m\operatorname{tr}(\boldsymbol{Y}_1^2 + \boldsymbol{Y}_2^2) \\ & +2\operatorname{tr}(\boldsymbol{Y}_1^3 + \boldsymbol{Y}_2^3) - 6a\operatorname{tr}(\boldsymbol{Y}_1\boldsymbol{Y}_2)] + O(a_3^{-3}).\end{aligned} \tag{13}$$

Therefore, using (12) and (13), we get

$$\ln \tilde{I}_1 + \ln \tilde{I}_2 = -\operatorname{tr}\,(\boldsymbol{Y}_1 + \boldsymbol{Y}_2) + \frac{\tilde{c}_1}{2a_3} + \frac{\tilde{c}_2}{6a_3^2} + O(a_3^{-3}),$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are given in the Theorem 4.1. Hence we get

$$\tilde{I}_1\tilde{I}_2 = \operatorname{etr}\,(-\boldsymbol{Y}_1 - \boldsymbol{Y}_2)\left[1 + \frac{\tilde{c}_1}{2a_3} + \frac{3\tilde{c}_1^2 + 4\tilde{c}_2}{24a_3^2} + O(a_3^{-3})\right]. \tag{14}$$

Finally, substituting from (14) in (11) we get the desired result. □

The expression (10) may be used to yield a corresponding asymptotic formula for the c.d.f. of $(\boldsymbol{W}_1, \boldsymbol{W}_2)$, *i.e.*,

$$P_2(\boldsymbol{A}_1, \boldsymbol{A}_2; a_1, a_2; a_3) = P_2(\boldsymbol{0} < \boldsymbol{W}_1 < \boldsymbol{A}_1, \boldsymbol{0} < \boldsymbol{W}_2 < \boldsymbol{A}_2).$$

Writing $\boldsymbol{B}_i = a_3\boldsymbol{A}_i$, $i = 1, 2$, we have

$$\begin{aligned}
&P_2(\boldsymbol{A}_1, \boldsymbol{A}_2; a_1, a_2; a_3)\\
&\quad = P_2(\boldsymbol{0} < \boldsymbol{Y}_1 < \boldsymbol{B}_1, \boldsymbol{0} < \boldsymbol{Y}_2 < \boldsymbol{B}_2)\\
&\quad = \int_{\boldsymbol{0}<\boldsymbol{Y}_2<\boldsymbol{B}_2}\int_{\boldsymbol{0}<\boldsymbol{Y}_1<\boldsymbol{B}_1}\left[\frac{\operatorname{etr}[-(\boldsymbol{Y}_1+\boldsymbol{Y}_2)]\,|\boldsymbol{Y}_1|^{a_1-m}\det(\boldsymbol{Y}_2)^{a_2-m}}{\tilde{\Gamma}_m(a_1)\tilde{\Gamma}_m(a_2)}\right]\\
&\qquad \times\left[1+\frac{\tilde{c}_1}{2a_3}+\frac{3\tilde{c}_1^2+4\tilde{c}_2}{24a_3^2}+O(a_3^{-3})\right]\mathrm{d}\boldsymbol{Y}_1\,\mathrm{d}\boldsymbol{Y}_2. \qquad (15)
\end{aligned}$$

It is seen that each term in (15) is a combination of the functions

$$\begin{aligned}
G_{\alpha,K_1,K_2}(\boldsymbol{B}_1, \boldsymbol{B}_2) = &\int_{\boldsymbol{0}<\boldsymbol{Y}_2<\boldsymbol{B}_2}\int_{\boldsymbol{0}<\boldsymbol{Y}_1<\boldsymbol{B}_1}\operatorname{etr}[-(\boldsymbol{Y}_1+\boldsymbol{Y}_2)]\,|\boldsymbol{Y}_1|^{a_1-m}\det(\boldsymbol{Y}_2)^{a_2-m}\\
&\times[\operatorname{tr}\{-(\boldsymbol{Y}_1+\boldsymbol{Y}_2)\}^{\alpha}]^{K_1}\,[\operatorname{tr}\{-(\boldsymbol{Y}_1+\boldsymbol{Y}_2)\}]^{K_2}\,\mathrm{d}\boldsymbol{Y}_1\,\mathrm{d}\boldsymbol{Y}_2. \qquad (16)
\end{aligned}$$

The integral on the right-hand side of (16) does not seem to be easy to evaluate. Further work on this is needed to be done.

Appendix: Additional definitions and results

We give a brief review of some definitions and notations. We adhere to standard notations (cf. Anderson et al. [1]). For a given $\boldsymbol{A} \in \mathbb{C}^{m\times m}$, $\boldsymbol{A}'$ denotes the transpose of $\boldsymbol{A}$, $\bar{\boldsymbol{A}}$ denotes the conjugate of $\boldsymbol{A}$, $\boldsymbol{A}^H$ denotes the conjugate transpose of $\boldsymbol{A}$, $\operatorname{tr}(\boldsymbol{A}) = a_{11} + \cdots + a_{mm}$; $\operatorname{etr}(\boldsymbol{A}) = \exp(\operatorname{tr}(\boldsymbol{A}))$; $|\boldsymbol{A}| =$ determinant of $\boldsymbol{A}$; norm of $\boldsymbol{A} = \|\boldsymbol{A}\| =$ maximum of absolute values of latent roots of the matrix $\boldsymbol{A}$; $\boldsymbol{A} = \boldsymbol{A}^H > \boldsymbol{0}$ means that $\boldsymbol{A}$ is a Hermitian positive definite, and $\boldsymbol{A}^{\frac{1}{2}}$ denotes the unique Hermitian positive definite square root of $\boldsymbol{A} = \boldsymbol{A}^H > \boldsymbol{0}$.

Let $\boldsymbol{A}$ be an $m \times n$ complex matrix of rank m ($m \leqslant n$) and let $\boldsymbol{B}$ be the Hermitian positive definite square root of $\boldsymbol{A}\boldsymbol{A}^H$ so that $\boldsymbol{A}\boldsymbol{A}^H = \boldsymbol{B}^2$. Then, it is well known that the matrix $\boldsymbol{A}$ can be written as $\boldsymbol{A} = \boldsymbol{B}\boldsymbol{G}_1$, where the $m \times n$ matrix $\boldsymbol{G}_1$ is semi-unitary, $\boldsymbol{G}_1\boldsymbol{G}_1^H = \boldsymbol{I}_m$. Further, choose an $(n-m) \times n$ matrix $\boldsymbol{G}_2$ such that the $n \times n$ matrix $\boldsymbol{G} = \begin{pmatrix}\boldsymbol{G}_1\\ \boldsymbol{G}_2\end{pmatrix}$ is unitary. Now, we can write $\boldsymbol{A} = \boldsymbol{B}\begin{pmatrix}\boldsymbol{I}_m & \boldsymbol{0}\end{pmatrix}\begin{pmatrix}\boldsymbol{G}_1\\ \boldsymbol{G}_2\end{pmatrix} = \boldsymbol{B}\begin{pmatrix}\boldsymbol{I}_m & \boldsymbol{0}\end{pmatrix}\boldsymbol{G}$.

The generalized hypergeometric function of Hermitian matrix argument is defined by

$$ {}_p\tilde{F}_q(a_1, \ldots, a_p; b_1, \ldots, b_q; \boldsymbol{X}) = \sum_{k=0}^{\infty}\sum_{\kappa\vdash k}\frac{[a_1]_\kappa\cdots[a_p]_\kappa}{[b_1]_\kappa\cdots[b_q]_\kappa}\frac{\tilde{C}_\kappa(\boldsymbol{X})}{k!}, \qquad (A.1)$$

where a_i, $i = 1, \ldots, p$, b_j, $j = 1, \ldots, q$ are arbitrary complex numbers, $\boldsymbol{X}$ ($m \times m$) is a Hermitian matrix, $\tilde{C}_\kappa(\boldsymbol{X})$ is the zonal polynomial of $m \times m$ Hermitian matrix $\boldsymbol{X}$ corresponding to the ordered partition $\kappa = (k_1, \ldots, k_m)$, $k_1 \geqslant \cdots \geqslant k_m \geqslant 0$, $k_1 + \cdots + k_m = k$ and $\sum_{\kappa\vdash k}$ denotes summation over all partitions κ. The complex multivariate hypergeometric coefficient $[a]_\kappa$ used above is defined by

$$[a]_\kappa = \prod_{i=1}^{m}(a-i+1)_{k_i} \qquad (A.2)$$

where $(a)_r = a(a+1)\cdots(a+r-1)$, $r = 1, 2, \ldots$ with $(a)_0 = 1$. Conditions for convergence of the series in (A.1) are available in the literature. From (A.1) it follows that

$$ {}_1\tilde{F}_0(a; \boldsymbol{X}) = \sum_{k=0}^{\infty}\sum_{\kappa\vdash k}\frac{[a]_\kappa\tilde{C}_\kappa(\boldsymbol{X})}{k!} = |\boldsymbol{I}_m - \boldsymbol{X}|^{-a}, \quad \|\boldsymbol{X}\| < 1, \qquad (A.3)$$

and

$$_2\tilde{F}_1(a, b; c; \boldsymbol{X}) = \sum_{k=0}^{\infty} \sum_{\kappa \vdash k} \frac{[a]_\kappa [b]_\kappa}{[c]_\kappa} \frac{\tilde{C}_\kappa(\boldsymbol{X})}{k!}, \quad \|\boldsymbol{X}\| < 1. \tag{A.4}$$

The integral representation of the Gauss hypergeometric function $_2F_1$ of Hermitian matrix is given by

$$_2\tilde{F}_1(a, b; c; \boldsymbol{X}) = \frac{1}{\tilde{B}_m(a, c-a)} \int_{\boldsymbol{0}<\boldsymbol{R}=\boldsymbol{R}^H<\boldsymbol{I}_m} \frac{|\boldsymbol{R}|^{a-m}|\boldsymbol{I}_m - \boldsymbol{R}|^{c-a-m}}{|\boldsymbol{I}_m - \boldsymbol{X}\boldsymbol{R}|^b} \, \mathrm{d}\boldsymbol{R}, \tag{A.5}$$

where $\mathrm{Re}(a, c-a) > m-1$. For $\mathrm{Re}(\alpha, \beta) > m-1$, we have

$$\int_{\boldsymbol{0}<\boldsymbol{R}=\boldsymbol{R}^H<\boldsymbol{I}_m} |\boldsymbol{R}|^{\alpha-m}|\boldsymbol{I}_m - \boldsymbol{R}|^{\beta-m}\tilde{C}_\kappa(\boldsymbol{X}\boldsymbol{R}) \, \mathrm{d}\boldsymbol{R} = \frac{\tilde{B}_m(\alpha, \beta)[\alpha]_\kappa}{[\alpha+\beta]_\kappa}\tilde{C}_\kappa(\boldsymbol{X}) \tag{A.6}$$

and

$$\begin{aligned} &\int_{\boldsymbol{0}<\boldsymbol{R}=\boldsymbol{R}^H<\boldsymbol{I}_m} |\boldsymbol{R}|^{\alpha-m}|\boldsymbol{I}_m - \boldsymbol{R}|^{\beta-m} \, _2\tilde{F}_1(a, b; c; \boldsymbol{X}\boldsymbol{R}) \, \mathrm{d}\boldsymbol{R} \\ &\quad = \tilde{B}_m(\alpha, \beta) \, _3\tilde{F}_2(\alpha, a, b; \alpha+\beta, c; \boldsymbol{X}). \end{aligned} \tag{A.7}$$

For properties and further results on these functions the reader is referred to James [6] and Chikuse [5]. Corresponding to the partitions κ and τ of k and t, respectively, the product $\tilde{C}_\kappa(\boldsymbol{X})\tilde{C}_\tau(\boldsymbol{X})$ is defined as (Chikuse [5]),

$$\tilde{C}_\kappa(\boldsymbol{W})\tilde{C}_\tau(\boldsymbol{W}) = \sum_{\delta \vdash d} \tilde{g}^\delta_{\kappa,\tau}\tilde{C}_\delta(\boldsymbol{W}), \tag{A.8}$$

where δ is the partition of the integer $d = k + t$ and $\tilde{g}^\delta_{\kappa,\tau}$ is the coefficient of $\tilde{C}_\delta(\boldsymbol{W})$ in $\tilde{C}_\kappa(\boldsymbol{W})\tilde{C}_\tau(\boldsymbol{W})$.

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